Open Mapping and Inverse Function Theorem

Note Title 4/30/2008

1. Let f be a nonconstant analytic function. Then if Ω is open $f(\Omega)$ is open.

pf: Let $z_0 \in \Omega$ and $f(z_0) = w_0 \in f(\Omega)$. Let $D_r = \{|z - z_0| \le r\} \subset \Omega$ be such that $f(z) \ne w_0$ if $z \in D_r - \{z_0\}$. Let $C_r = \{|z - z_0| = r\} = \partial D_r$. Then $\delta = \text{dist}(w_0, f(C_r)) > 0$. Let w be such that $|w - w_0| < \delta$. If $z \in C_r$, $|f(z) - w_0 - (f(z) - w)| = |w - w_0|$

$|w - w_0| < \delta \le |f(z) - w_0| \le |f(z - w_0)| + |f(z) - w|$,

so by Rouché $f(z) - w_0$ and $f(z) - w)$ have the same number of zeros in the interior of D_r.

Hence there is a z with $f(z) = w$. This proves $\{|w - w_0| < \delta\} \subset f(\Omega)$, so $f(\Omega)$ is open.

2. If $f'(z_0) \ne 0$, the order of the zero of $f(z) - w_0$ is 1. So given w with $|w - w_0| < \delta$, there is a unique z, with $|z - z_0| < r$ so that $f(z) = w$. Thus f has an inverse $F(w)$ on $\{|w - w_0| < \delta\}$. By the residue theorem

$$F(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{\zeta f'(\zeta)\, d\zeta}{(f(\zeta) - w)}$$

Hence F is analytic.